Actions of quantum groups on operator systems

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Joint work with Lucas Hataishi

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GOAL AND CONTENT

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3. Quantum groups
4. Actions of quantum groups
5. Injectivity of crossed products

OPERATOR SYSTEMS AND TENSOR PRODUCTS

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- We define

$$X \otimes Y = \overline{\text{span}}^{\|\cdot\|}\{x \otimes y : x \in X, y \in Y\} \subseteq B(\mathcal{H} \otimes \mathcal{K})$$

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- We define the Fubini tensor product $X \overline{\otimes} Y$ to be the set of all $z \in B(\mathcal{H} \otimes \mathcal{K})$ such that $(\omega \overline{\otimes} \text{id})(z) \in Y$ for all $\omega \in B(\mathcal{H})_*$ and $(\text{id} \overline{\otimes} \chi)(z) \in X$ for all $\chi \in B(\mathcal{K})_*$.

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Fubini crossed product (Hamana, 1980)

The set

$$A \rtimes_{\alpha,\mathcal{F}} \Gamma := \{z \in A \overline{\otimes} B(\ell^2(\Gamma)) : \forall g, h, k \in \Gamma : \beta_{g^{-1}}(z_{h,k}) = z_{hg,kg}\}$$

is called the Fubini crossed product.

Injectivity of Fubini crossed product (Hamana, 1981)

The following statements are equivalent:

- A is Γ-injective.
- $A \rtimes_{\alpha,\mathcal{F}} \Gamma$ is injective.

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  - $L(\Gamma)$, the left group von Neumann algebra, is injective.
- **Goal**: Find suitable generalisation when the discrete group Γ is replaced by a discrete quantum group.

COMPACT QUANTUM GROUPS

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- Let $\mathbb{G}$ be a CQG (in the sense of Woronowicz) with function algebra $(C(\mathbb{G}), \Delta_{\mathbb{G}})$ and Haar state $\varphi_{\mathbb{G}} : C(\mathbb{G}) \to \mathbb{C}$ with GNS-construction $(L^2(\mathbb{G}), \lambda, \xi_{\mathbb{G}})$.

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- Let $\mathcal{O}(\mathbb{G})$ be the dense Hopf $*$-subalgebra of $C(\mathbb{G})$.
- Consider the multiplicative unitaries $V_{\mathbb{G}}$ and $W_{\mathbb{G}}$ given by

$$V_{\mathbb{G}}(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \odot \Lambda)(\Delta(a)(1 \otimes b)),$$
$$W_{\mathbb{G}}^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \odot \Lambda)(\Delta(b)(a \otimes 1)).$$

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- We define $C_r(\mathbb{G}) = \lambda(C(\mathbb{G}))$ and $\mathscr{L}^\infty(\mathbb{G}) := C_r(\mathbb{G})''$ and we endow them with their usual coproduct

$$\Delta(x) = V_{\mathbb{G}}(x \otimes 1)V_{\mathbb{G}}^*, \quad x \in B(L^2(\mathbb{G})).$$

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- Consider the multiplier Hopf *-algebra $(c_c(\widehat{\mathbb{G}}), \widetilde{\Delta})$ dual to $(\mathcal{O}(\mathbb{G}), \Delta)$, i.e.

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- Define dual GNS-maps $\hat{\Gamma}, \hat{\Lambda} : c_c(\widehat{\mathbb{G}}) \to B(L^2(\mathbb{G}))$ by

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- We also define $\hat{\lambda} : c_c(\widehat{\mathbb{G}}) \to B(L^2(\mathbb{G}))$ by

$$\hat{\lambda}(\omega)\hat{\Gamma}(\chi) := \hat{\Gamma}(\omega \star \chi)$$

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$$\hat{\Delta}_l : B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G}))\overline{\otimes}B(L^2(\mathbb{G})) : x \mapsto W_{\widehat{\mathbb{G}}}^*(1 \otimes x)W_{\widehat{\mathbb{G}}}$$
$$\hat{\Delta}_r : B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G}))\overline{\otimes}B(L^2(\mathbb{G})) : x \mapsto V_{\widehat{\mathbb{G}}}(x \otimes 1)V_{\widehat{\mathbb{G}}}^*$$

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- These comultiplications agree on $\mathscr{L}^\infty(\widehat{\mathbb{G}})$: $\hat{\Delta} := \hat{\Delta}_l = \hat{\Delta}_r$.

ACTIONS OF COMPACT QUANTUM GROUPS

$\mathbb{G}$-C^*-actions

A right $\mathbb{G}$-C^*-operator system is a pair (X, α) where X is an operator system and $\alpha : X \to X \otimes C_r(\mathbb{G})$ is a uci map such that:

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- The usual spectral decomposition works.
- If X is a C^*-algebra, then α is automatically multiplicative!

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A right $\mathbb{G}$-W^*-operator system is a pair (X, α) where X is an operator system and $\alpha : X \to X \overline{\otimes} \mathscr{L}^\infty(\mathbb{G})$ is a uci map such that the coaction property is satisfied, i.e. $(\alpha \overline{\otimes} \mathrm{id})\alpha = (\mathrm{id} \overline{\otimes} \Delta)\alpha$. We will write $X \overset{\alpha}{\curvearrowleft} \mathbb{G}$ and say that α defines a right $\mathbb{G}$-W^*-action on the operator system X.

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Regular elements

If (X, α) is a $\mathbb{G}$-W^*-operator system, we define the algebraic regular elements and the regular elements by

$$\mathcal{R}_{\mathrm{alg}}(X, \alpha) := \{x \in X : \alpha(x) \in X \odot \mathcal{O}(\mathbb{G})\},$$

$$\mathcal{R}(X, \alpha) := \overline{\mathcal{R}_{\mathrm{alg}}(X, \alpha)}^{\|\cdot\|}.$$

EQUIVARIANT MAPS

$\mathbb{G}$-C^* and $\mathbb{G}$-W^*-equivariant maps

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$\mathbb{G}$-C^* and $\mathbb{G}$-W^*-equivariant maps

- Let (X, α) and (Y, β) two $\mathbb{G}$-C^*-operator systems. A ucp map $\phi : X \to Y$ is called $\mathbb{G}$-C^*-equivariant if the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \downarrow{\alpha} & & \downarrow{\beta} \\ X \otimes C_r(\mathbb{G}) & \xrightarrow{\phi \otimes \mathrm{id}} & Y \otimes C_r(\mathbb{G}) \end{array}$$

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- A $\mathbb{G}$-C^*-operator system I is called $\mathbb{G}$-C^*-injective if for all $\mathbb{G}$-C^*-operator systems X, Y, every $\mathbb{G}$-C^*-ucp map $\varphi : X \to I$ and every $\mathbb{G}$-C^*-uci map $\iota : X \to Y$, there exists a $\mathbb{G}$-C^*-ucp map $\Phi : Y \to I$ such that $\Phi\iota = \phi$.

$$\begin{array}{ccc} X & \xrightarrow{\quad\phi\quad} & I \\ {\scriptstyle\iota}\big\downarrow & \nearrow_{\exists\Phi} & \\ Y & & \end{array}$$

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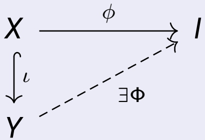

- Similarly, $\mathbb{G}$-W^*-injectivity is defined.

$\mathbb{G}$-EQUIVARIANT INJECTIVITY

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Elementary properties

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- If X is injective as an operator system, then $(X \overline{\otimes} B(L^2(\mathbb{G})), \mathrm{id}_X \overline{\otimes} \Delta)$ is $\mathbb{G}$-W^*-injective.

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  1. (X, α) is $\mathbb{G}$-W^*-injective.
  2. X is injective and there exists a $\mathbb{G}$-W^* ucp conditional expectation $\varphi : (X \overline{\otimes} B(L^2(\mathbb{G})), \mathrm{id} \overline{\otimes} \Delta) \to (\alpha(X), \mathrm{id} \overline{\otimes} \Delta)$.

ACTIONS OF DISCRETE QUANTUM GROUPS

$\widehat{\mathbb{G}}$-actions

A (right) $\widehat{\mathbb{G}}$-operator system is a pair (X, α) where X is an operator system and $\alpha : X \to X \overline{\otimes} \mathscr{L}^\infty(\widehat{\mathbb{G}})$ is a uci map such that $(\alpha \overline{\otimes} \mathrm{id})\alpha = (\mathrm{id} \overline{\otimes} \hat{\Delta})\alpha$. We will write $X \overset{\alpha}{\curvearrowleft} \widehat{\mathbb{G}}$ and say that α defines a right $\widehat{\mathbb{G}}$-action on the operator system X.

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- If X is a C^*-algebra, then $X \overline{\otimes} \mathscr{L}^\infty(\widehat{\mathbb{G}}) \cong \prod_{\pi \in \mathrm{Irr}(\mathbb{G})} (X \otimes B(\mathcal{H}_\pi))$ carries a natural C^*-algebra structure. Then α is a $*$-homomorphism (result by S. Vaes).

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- Equivalent description in terms of $\mathcal{O}(\mathbb{G})$-module operator systems.

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- Equivalent description in terms of $\mathcal{O}(\mathbb{G})$-module operator systems.
- Notion of $\widehat{\mathbb{G}}$-equivariant map and $\widehat{\mathbb{G}}$-injective operator system defined in the obvious way.

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- Generalise classical crossed products.
- If $X \overset{\tau}{\curvearrowleft} \widehat{\mathbb{G}}$ trivially, then $X \rtimes_{\tau,r} \widehat{\mathbb{G}} = X \otimes C_r(\mathbb{G})$ and $X \rtimes_{\tau,\mathcal{F}} \widehat{\mathbb{G}} = X \overline{\otimes} \mathscr{L}^\infty(\mathbb{G})$.

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- We can still repair the statement by asking for equivariant injectivity of the crossed products!

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[Modulo technical details]

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Cotensor product of operator systems

Suppose that X, Y, Z are operator systems and $\beta_r : X \to X \overline{\otimes} Z$ and $\beta_l : Y \to Z \overline{\otimes} Y$ ucp maps. We define the cotensor product

$$X \boxtimes_{(\beta_l, \beta_r)} Y := \{\xi \in X \overline{\otimes} Y : (\mathrm{id}_X \overline{\otimes} \beta_l)(\xi) = (\beta_r \overline{\otimes}\, \mathrm{id}_Y)(\xi)\}.$$

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If S is an operator system and $\delta : Y \to Y\overline{\otimes}S$ is a ucp map such that

$$(\mathrm{id}_Z \overline{\otimes}\delta) \circ \beta_l = (\beta_l\overline{\otimes}\,\mathrm{id}_S) \circ \delta$$

then the map $\mathrm{id}_X \overline{\otimes}\delta : X\overline{\otimes}Y \to X\overline{\otimes}Y\overline{\otimes}S$ restricts to a map $X \boxtimes_{(\beta_l,\beta_r)} Y \to (X \boxtimes_{(\beta_l,\beta_r)} Y)\overline{\otimes}S$.

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- We can apply this to the Fubini crossed product $X \rtimes_{\alpha,\mathcal{F}} \widehat{\mathbb{G}} = X \boxtimes_{(\hat{\Delta}_l,\alpha)} B(L^2(\mathbb{G}))$ with $\delta = \Delta : B(L^2(\mathbb{G})) \to B(L^2(\mathbb{G}))\overline{\otimes}\mathscr{L}^\infty(\mathbb{G})$.

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In particular, considering $X = \mathbb{C}$, we find that the following statements are equivalent:

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We sketch the proof. We need two main ingredients first.

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Regular elements Fubini crossed product

Let (X, α) be a $\widehat{\mathbb{G}}$-operator system. Then
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$$z = \sum_{\pi \in \mathrm{Irr}(\mathbb{G})} \sum_{i,j=1}^{n_\pi} z_{ij}^\pi (1 \otimes u_{ij}^\pi), \quad z_{ij}^\pi \in X \overline{\otimes} \mathscr{L}^\infty(\widehat{\mathbb{G}}).$$

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- Show that $z_{ij}^\pi \in X \rtimes_{\alpha,\mathcal{F}} \widehat{\mathbb{G}}$. Conclude that $z_{ij}^\pi \in \alpha(X)$.

□

INGREDIENT 2

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Automatic invariance

Let $X \subseteq B(\mathcal{H})$ be an operator system and

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a ucp map such that $\psi(1 \otimes a) = 1 \otimes a$ for all $a \in \mathcal{O}(\mathbb{G})$. Then ψ automatically preserves the right coaction

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Proof.

Multiplicative domain argument. □

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- Define $\psi := (\phi \overline{\otimes} \mathrm{id}) \circ (\mathrm{id} \overline{\otimes} \hat{\Delta}_l) : X \overline{\otimes} B(L^2(\mathbb{G})) \to X \overline{\otimes} B(L^2(\mathbb{G}))$.

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- X is injective since there is a $\mathbb{G}$-C^*-conditional expectation $E : (X \rtimes_{\alpha,r} \widehat{\mathbb{G}}, \mathrm{id} \overline{\otimes} \Delta) \to (X, \tau)$.

□

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Let X be a $\mathbb{G}$-C^*-operator system. There exists a $\mathbb{G}$-C^*-injective envelope (S, ι) for X. If $(\widetilde{S}, \tilde{\iota})$ is another $\mathbb{G}$-C^*-injective envelope, there exists a unique $\mathbb{G}$-C^*-unital order isomorphism $\theta : S \to \widetilde{S}$ such that $\theta \circ \iota = \widetilde{\iota}$. commutes. Moreover, $(S, \iota : X \to S)$ is a $\mathbb{G}$-C^*-injective envelope if and only if (S, ι) is $\mathbb{G}$-C^*-injective and $\mathbb{G}$-C^*-rigid.

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EQUIVARIANT INJECTIVE ENVELOPES

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- Consider the set $\mathcal{G}$ of $\mathbb{G}$-C^*-ucp maps $\phi : Y \to Y$ that satisfy $\phi \circ \alpha = \alpha$.
- Choose a minimal idempotent $\phi_0 \in \mathcal{G}$ and check that $\phi_0(Y)$ is a $\mathbb{G}$-C^*-injective envelope.

□

DUALITY OF EQUIVARIANT INJECTIVE ENVELOPES

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$\mathbb{G}$-C^*-Rigidity of reduced crossed products

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2. $(Y \rtimes_{r,\beta} \widehat{\mathbb{G}}, \iota \rtimes_r \widehat{\mathbb{G}} : (X \rtimes_{r,\alpha} \widehat{\mathbb{G}}, \mathrm{id} \overline{\otimes} \Delta) \to (Y \rtimes_{r,\beta} \widehat{\mathbb{G}}, \mathrm{id} \overline{\otimes} \Delta))$ is a $\mathbb{G}$-C^*-rigid extension of $X \rtimes_{r,\alpha} \widehat{\mathbb{G}}$.

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Equivariant injective envelopes and crossed products

Let X be a $\widehat{\mathbb{G}}$-operator system. Let (Y, ι) be a $\widehat{\mathbb{G}}$-extension of X. Then (Y, ι) is the $\widehat{\mathbb{G}}$-injective envelope of X if and only if $(Y \rtimes_r \widehat{\mathbb{G}}, \iota \rtimes_r \widehat{\mathbb{G}})$ is the $\mathbb{G}$-C^*-injective envelope of $X \rtimes_r \widehat{\mathbb{G}}$. In particular,

$$I_{\widehat{\mathbb{G}}}(X) \rtimes_r \widehat{\mathbb{G}} = I_{\mathbb{G}}^{C^*}(X \rtimes_r \widehat{\mathbb{G}}).$$

RELEVANT LITERATURE

Joeri De Ro and Lucas Hataishi

"Actions of compact and discrete quantum groups on operator systems"

Erik Habbestad, Lucas Hataishi, and Sergey Neshveyev

"Noncommutative Poisson boundaries and Furstenberg-Hamana boundaries of Drinfeld doubles"

Masamichi Hamana

"Tensor products for monotone complete C^*-algebras. I, II"

Mehrdad Kalantar, Paweł Kasprzak, Adam Skalski and Roland Vergnioux

"Noncommutative Furstenberg boundary"